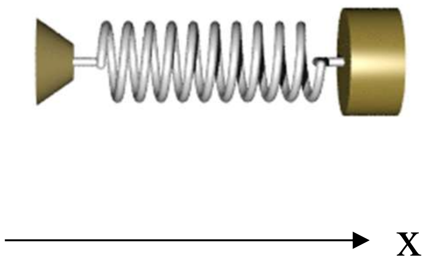
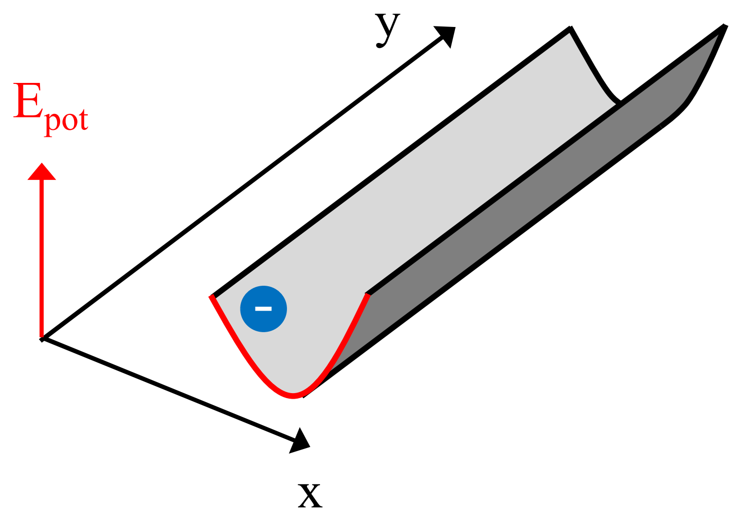
Exercice 7.1 : Lois de conservation:

Si un opérateur A ne dépend pas explicitement du temps, sa valeur moyenne est conservée si A commute avec l'hamiltonien H

Exercice:

Supposons: $$H = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}\kappa \cdot x^2$$





Lesquelles de ces valeurs moyennes sont conservées ?

$\langle X \rangle$

$\langle Y \rangle$

$\langle P_x \rangle$

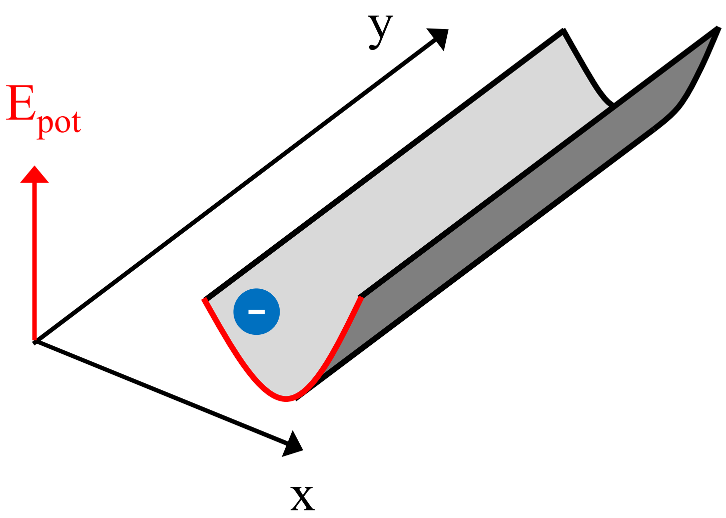
$\langle P_y \rangle$

Exercice 7.1 : Lois de conservation:

Théorème d'Ehrenfest: $$\frac{\partial}{\partial t}\langle A\rangle = \left\langle \frac{\partial A}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle [A,H]\rangle$$

Exercice:

Supposons: $$H = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}\kappa \cdot x^2$$



Lesquelles de ces valeurs moyennes sont conservées ?

$$[X, P_x] \equiv i\hbar \quad \Rightarrow \quad \cancel{\langle X\rangle}$$

$$[Y, P_y] \equiv i\hbar \quad \Rightarrow \quad \cancel{\langle Y\rangle}$$

$$[X, P_x] \equiv i\hbar \quad \Rightarrow \quad \cancel{\langle P_x\rangle}$$

$$\boxed{\langle P_y\rangle = \text{conservée}}$$

Exercice 7.2: évolution des opérateurs

Soit un Hamiltonien de la forme:

$$H = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + V\left(\vec{x}\right)$$

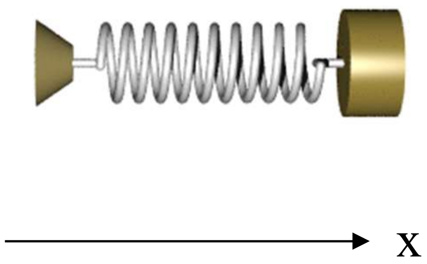
Calculez les évolutions suivantes:

$$\frac{\partial}{\partial t}\left\langle \vec{X} \right\rangle$$

$$\frac{\partial}{\partial t}\left\langle \vec{P} \right\rangle$$

appliquez-le avec l'Hamiltonien 1D suivant:

$$H = \frac{P_x^2}{2m} + \frac{1}{2}\kappa \cdot x^2$$



Exercice 7.2: évolution des opérateurs

$$\left[X,P_x^2\right]=XP_x\cdot P_x-P_xP_x\cdot X=XP_x\cdot P_x-P_xX\cdot P_x+P_x\cdot XP_x-P_x\cdot P_xX$$

$$=\left[X,P_x\right]\cdot P_x+P_x\cdot\left[X,P_x\right]=i\hbar\cdot P_x+P_x\cdot i\hbar=2i\hbar\cdot P_x$$

$$H=\frac{P_x^2+P_y^2+P_z^2}{2m}+V\left(\vec{x}\right)$$

$$\left.\begin{aligned}
\frac{\partial}{\partial t}\langle X\rangle&=\cancel{\left\langle\frac{\partial X}{\partial t}\right\rangle}+\frac{1}{i\hbar}\langle[X,H]\rangle=0+\frac{1}{i\hbar}\left\langle\left[X,\frac{P_x^2}{2m}\right]\right\rangle=\frac{\langle P_x\rangle}{m}\\
\frac{\partial}{\partial t}\langle Y\rangle&=\cancel{\left\langle\frac{\partial Y}{\partial t}\right\rangle}+\frac{1}{i\hbar}\langle[Y,H]\rangle=0+\frac{1}{i\hbar}\left\langle\left[Y,\frac{P_y^2}{2m}\right]\right\rangle=\frac{\langle P_y\rangle}{m}\\
\frac{\partial}{\partial t}\langle Z\rangle&=\cancel{\left\langle\frac{\partial Z}{\partial t}\right\rangle}+\frac{1}{i\hbar}\langle[Z,H]\rangle=0+\frac{1}{i\hbar}\left\langle\left[Z,\frac{P_z^2}{2m}\right]\right\rangle=\frac{\langle P_z\rangle}{m}
\end{aligned}\right\}$$

$$\boxed{m\cdot\frac{\partial}{\partial t}\left\langle\vec{X}\right\rangle=\left\langle\vec{P}\right\rangle}$$

$$m\cdot\vec{v}=\vec{p}$$

Exercice 7.2: évolution des opérateurs

$$P_x V\cdot|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\left(V|\psi\rangle\right) = -i\hbar\left(\frac{\partial}{\partial x}V\right)\cdot|\psi\rangle + V\cdot\left(-i\hbar\frac{\partial}{\partial x}\right)|\psi\rangle = -i\hbar\left(\frac{\partial}{\partial x}V\right)\cdot|\psi\rangle + VP_x\cdot|\psi\rangle \Rightarrow \boxed{[P_x,V] = -i\hbar\left(\frac{\partial}{\partial x}V\right)}$$

$$H = \frac{P_x^2+P_y^2+P_z^2}{2m} + V(\vec{x})$$

$$\left.\begin{aligned}
\frac{\partial}{\partial t}\langle P_x\rangle &= \left\langle\cancel{\frac{\partial P_x}{\partial t}}\right\rangle + \frac{1}{i\hbar}\langle[P_x,H]\rangle = 0 + \frac{1}{i\hbar}\langle[P_x,V]\rangle = \left\langle -\frac{\partial}{\partial x}V\right\rangle \\
\frac{\partial}{\partial t}\langle P_y\rangle &= \left\langle\cancel{\frac{\partial P_y}{\partial t}}\right\rangle + \frac{1}{i\hbar}\langle[P_y,H]\rangle = 0 + \frac{1}{i\hbar}\langle[P_y,V]\rangle = \left\langle -\frac{\partial}{\partial y}V\right\rangle \\
\frac{\partial}{\partial t}\langle P_z\rangle &= \left\langle\cancel{\frac{\partial P_z}{\partial t}}\right\rangle + \frac{1}{i\hbar}\langle[P_z,H]\rangle = 0 + \frac{1}{i\hbar}\langle[P_z,V]\rangle = \left\langle -\frac{\partial}{\partial z}V\right\rangle
\end{aligned}\right\}$$

$$\frac{\partial}{\partial t}\langle\vec{P}\rangle = \langle-\vec{\nabla}V(\vec{x})\rangle \equiv \langle\vec{F}\rangle$$

$$\boxed{m\cdot\frac{\partial^2}{\partial t^2}\langle\vec{X}\rangle = \langle\vec{F}\rangle}$$

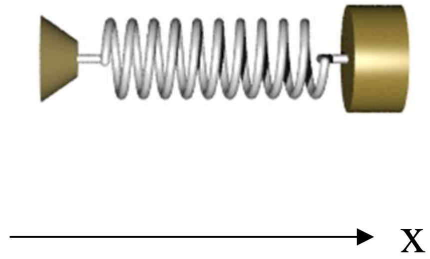
$$m\cdot\vec{a} = \vec{F}$$

PAB

Exercice 7.2: évolution des opérateurs

appliquez avec l'Hamiltonien

$$H = \frac{P_x^2}{2m} + \frac{1}{2}\kappa \cdot x^2$$



$$F_x = -\frac{\partial}{\partial x} V(x) = -\kappa \cdot x$$

$$A = \langle x \rangle (0)$$

$$\frac{\partial}{\partial t}\langle P_x \rangle = \langle F_x \rangle \qquad \frac{\partial}{\partial t}\langle x \rangle = \frac{\langle P_x \rangle}{m}$$

$$B = \frac{1}{\sqrt{m \cdot \kappa}} \cdot \langle P_x \rangle (0)$$

$$\boxed{\frac{\partial^2}{\partial t^2}\langle x \rangle = \frac{1}{m}\frac{\partial}{\partial t}\langle P_x \rangle = -\frac{\kappa}{m}\langle x \rangle \qquad \Rightarrow \qquad \langle x \rangle = A \cdot \cos\left(\sqrt{\frac{\kappa}{m}} \cdot t\right) + B \cdot \sin\left(\sqrt{\frac{\kappa}{m}} \cdot t\right)}$$

$$\boxed{\langle P_x \rangle = m \cdot \frac{\partial}{\partial t}\langle x \rangle \qquad \Rightarrow \qquad \langle P_x \rangle = \sqrt{m \cdot \kappa} \cdot \left(-A \cdot \sin\left(\sqrt{\frac{\kappa}{m}} \cdot t\right) + B \cdot \cos\left(\sqrt{\frac{\kappa}{m}} \cdot t\right)\right)}$$

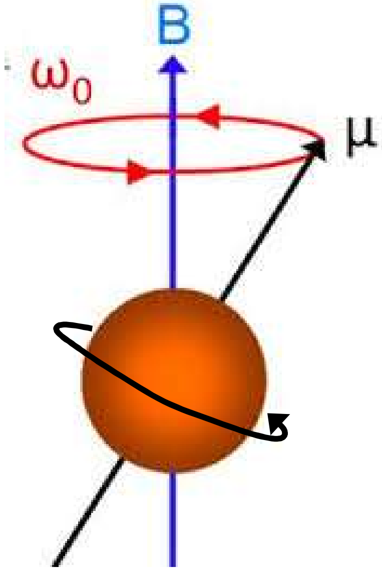
Exercice 7.3

$$H = -\vec{\mu}\cdot\vec{B} = -\frac{\hbar}{2}\gamma\left(B_x\sigma_X + B_y\sigma_Y + B_z\sigma_Z\right) \qquad \text{avec} \qquad \vec{B} = \begin{pmatrix} 0 \\ 0 \\ B_Z \end{pmatrix}$$

$$\left[\sigma_X,\sigma_Y\right] \equiv 2i\cdot\sigma_Z$$

$$\left[\sigma_Y,\sigma_Z\right] \equiv 2i\cdot\sigma_X$$

$$\left[\sigma_Z,\sigma_X\right] \equiv 2i\cdot\sigma_Y$$



$\vec{\mu} = \gamma\frac{\hbar}{2}\cdot\vec{\sigma}$

Moment magnétique

1) Montrez que $\langle\sigma_z\rangle$ est conservé

2) Au temps t=0, $\langle\sigma_x(0)\rangle = 1 \quad et \quad \langle\sigma_y(0)\rangle = 0$

Calculez l'évolution de $\langle\sigma_x(t)\rangle \, et \, \langle\sigma_y(t)\rangle$

Base de la NMR
et de l'IRM

Exercice 7.3

$B_x=0$ et $B_y=0$ $\Longrightarrow$ $H=-\frac{\hbar}{2}\gamma\left(B_z\sigma_Z\right)$

$\left[\sigma_X,\sigma_Y\right]\equiv 2i\cdot\sigma_Z$ $\qquad$ $\left[\sigma_Y,\sigma_Z\right]\equiv 2i\cdot\sigma_X$ $\qquad$ $\left[\sigma_Z,\sigma_X\right]\equiv 2i\cdot\sigma_Y$

$$\frac{\partial}{\partial t}\left\langle\sigma_X\right\rangle=\frac{1}{i\hbar}\left\langle\left[\sigma_x,H\right]\right\rangle=\frac{-\hbar\gamma}{2i\hbar}\left\langle\left[\sigma_x,\sigma_z\right]\right\rangle B_z=\gamma\left\langle\sigma_y\right\rangle B_z\neq 0 \quad\Longrightarrow\quad \left\langle\sigma_x\right\rangle \text{ varie}$$

$$\frac{\partial}{\partial t}\left\langle\sigma_y\right\rangle=\frac{1}{i\hbar}\left\langle\left[\sigma_y,H\right]\right\rangle=\frac{-\hbar\gamma}{2i\hbar}\left\langle\left[\sigma_y,\sigma_z\right]\right\rangle B_z=-\gamma\left\langle\sigma_x\right\rangle B_z\neq 0 \quad\Longrightarrow\quad \left\langle\sigma_y\right\rangle \text{ varie}$$

$$\frac{\partial}{\partial t}\left\langle\sigma_z\right\rangle=\frac{1}{i\hbar}\left\langle\left[\sigma_z,H\right]\right\rangle=\frac{-\hbar\gamma}{2i\hbar}\left\langle\left[\sigma_z,\sigma_z\right]\right\rangle B_z=0 \quad\Longrightarrow\quad \left\langle\sigma_z\right\rangle \textit{ conservé}$$

Exercice 7.3

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t}\langle\sigma_X\rangle = \gamma B_z\langle\sigma_y\rangle \\[2em] \dfrac{\partial}{\partial t}\langle\sigma_y\rangle = -\gamma B_z\langle\sigma_x\rangle\end{array}\right\} \Rightarrow \begin{array}{l}\dfrac{\partial^2}{\partial t^2}\langle\sigma_X\rangle = \gamma B_z\dfrac{\partial}{\partial t}\langle\sigma_y\rangle = -\Omega_L^2\cdot\langle\sigma_x\rangle \\[2em] \dfrac{\partial^2}{\partial t^2}\langle\sigma_y\rangle = -\gamma B_z\dfrac{\partial}{\partial t}\langle\sigma_x\rangle = -\Omega_L^2\cdot\langle\sigma_y\rangle\end{array}$$

Avec la fréquence de Larmor: $\Omega_L \equiv \gamma B_z$

$$\langle\sigma_x(0)\rangle = 1 \quad et \quad \langle\sigma_y(0)\rangle = 0 \quad \Longrightarrow$$

$$\langle\sigma_X(t)\rangle = \cos(\Omega_L t)$$

$$\langle\sigma_y(t)\rangle = -\sin(\Omega_L t)$$

$$\langle\sigma_z\rangle \quad conservé$$

Fréquence de Larmor